BTO Ping Ring Function

A *BTO Ping Ring* is the intersection of a sphere centred on the Inmarsat 3F1 satellite with an ellipsoid centred on earth. The radius of the sphere is the distance between the satellite and the aeroplane, derived from a BTO observation. The size of the earth-centred ellipsoid is determined by the height of the aeroplane above some reference ellipsoid representing the earth. The aeroplane height is measured along a normal to the surface of the reference ellipsoid. We adopt WGS 84 as the reference ellipsoid.

We seek the function $f\colon \mathbb{R}^5 \to \mathbb{R}$ mapping the latitude φ and height *h* (above a reference ellipsoid) of a BTO ring, the ECEF Cartesian satellite position vector (X,Y,Z), and the distance between the ring and the satellite (R), to the longitude λ in the codomain of the ring. The BTO ring is a closed curve on a latitude-longitude surface patch and is the intersection of the satellite-centred sphere with the earth-centred ellipsoid.

We start with the standard equation for an ellipsoid of revolution with semi-axes *a* and *b*,

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1. \quad (1)$$

noting that $1 - e^2 = \frac{b^2}{a^2}$, where *e* is the eccentricity of the ellipsoid, leads to the equivalent ellipsoid equation,

$$(1 - e^2)(x^2 + y^2 - a^2) + z^2 = 0. \quad (2)$$

Parameters *a* and *e* will later be redefined as functions of the semi-major axis and eccentricity of the reference ellipsoid, and of the height of the BTO ring above the reference ellipsoid.

The standard equation for a sphere of radius *R* and centre (X,Y,Z) is,

$$(x - X)^2 + (y - Y)^2 + (z - Z)^2 = R^2. \quad (3)$$

Solving (3) for x^2, y^2 and z^2 respectively gives,

$$x^2 = R^2 - (y - Y)^2 - (z - Z)^2 - X^2 + 2Xx \quad (4a)$$

$$y^2 = R^2 - (z - Z)^2 - (x - X)^2 - Y^2 + 2Yy \quad (4b)$$

$$z^2 = R^2 - (x - X)^2 - (y - Y)^2 - Z^2 + 2Zz\ . \quad (4c)$$

Using the right side of system (4) in (2) yields,

$$(1 - e^2)[(R^2 - (y - Y)^2 - (z - Z)^2 - X^2 + 2Xx) + (R^2 - (z - Z)^2 - (x - X)^2 - Y^2 + 2Yy) - a^2] + (R^2 - (x - X)^2 - (y - Y)^2 - Z^2 + 2Zz) = 0. \quad (5)$$

We re-write (5) as,

$$(e^2 - 2)(x^2 + y^2) + 2(e^2 - 1)z^2 - 2(2e^2 - 3)(Xx + Yy) - 2(2e^2 - 3)Zz + (2e^2 - 3)(X^2 + Y^2 + Z^2 - R^2) - (1 - e^2)a^2 = 0. \quad (6)$$

In (6) we deliberately leave the third and fourth terms with a common factor, as will be soon justified.

To ease readability of (6) three substitutions are introduced,

$$\sigma_1 = e^2 - 2 \tag{7a}$$

$$\sigma_2 = 2(2e^2 - 3) \tag{7b}$$

$$\sigma_3 = (2e^2 - 3)(X^2 + Y^2 + Z^2 - R^2) - (1 - e^2)a^2. \tag{7c}$$

Each substitution in (7) depends on the height of the BTO ring above the reference ellipsoid via the dependency of eccentricity *e* and semi-major axis *a* on this height. The third substitution depends additionally on the satellite position (X,Y,Z), and on distance *R* between the BTO ring and satellite.

Using (7) in (6) gives,

$$\sigma_1(x^2 + y^2) + 2(e^2 - 1)z^2 - \sigma_2(Xx + Yy) - \sigma_2 Zz + \sigma_3 = 0 \tag{8}$$

We should like next to switch from Cartesian (*x,y,z*) to geodetic coordinates (φ, λ) for which the following system is required,

$$x = N\cos\varphi\cos\lambda \tag{9a}$$

$$y = N\cos\varphi\sin\lambda \tag{9b}$$

$$z = N(1 - e^2)\sin\varphi \tag{9c}$$

Each element in (9) depends on *N*, a function of the aeroplane latitude φ and the parameters defining the aeroplane ellipsoid, *a* and *e*. These parameters are in fact functions of the semi-major axis and eccentricity of the reference ellipsoid together with the aeroplane height. That is,

$$N(\varphi, a, e) = \frac{a}{\sqrt{1 - e^2\sin^2\varphi}} \tag{10}$$

and, using subscript RE to denote the reference ellipsoid parameters,

$$a(a_{RE}, h) = a_{RE} + h, \quad b(b_{RE}, h) = b_{RE} + h, \quad e(a_{RE}, b_{RE}, h) = \sqrt{1 - \left(\frac{b_{RE} + h}{a_{RE} + h}\right)^2}. \tag{11}$$

Using (9) in (8), after some manipulations to isolate the circular functions involving the longitude λ of a point on the BTO ring yields,

$$X\cos\lambda + Y\sin\lambda = \frac{\sigma_1 N\cos^2\varphi - 2(1-e^2)^3 N\sin^2\varphi - \sigma_2 Z(1-e^2)\sin\varphi + \frac{\sigma_3}{N}}{\sigma_2\cos\varphi}. \tag{12}$$

An identity for the left side of (12), obtained from the harmonic addition theorem[1], is

$$X\cos\lambda + Y\sin\lambda \equiv \mathrm{sgn}(X)\sqrt{X^2 + Y^2}\cos\left(\lambda - \tan^{-1}\frac{Y}{X}\right), \tag{13}$$

[1] Weisstein, Eric W. "Harmonic Addition Theorem." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/HarmonicAdditionTheorem.html

permitting us to write (12) as,

$$\cos\left(\lambda-\tan^{-1}\frac{Y}{X}\right)=\frac{\sigma_1 N\cos^2\varphi-2\left(1-e^2\right)^3 N\sin^2\varphi-\sigma_2 Z\left(1-e^2\right)\sin\varphi+{}^{\sigma_3}/_{N}}{\operatorname{sgn}(X)\sigma_2\cos\varphi\sqrt{X^2+Y^2}}\;. \tag{14}$$

Finally, solving (14) for the longitude λ of a point on the eastern half of a BTO ring,

$$\lambda=\cos^{-1}\left[\frac{\sigma_1 N\cos^2\varphi-2\left(1-e^2\right)^3 N\sin^2\varphi-\sigma_2 Z\left(1-e^2\right)\sin\varphi+{}^{\sigma_3}/_{N}}{\operatorname{sgn}(X)\sigma_2\cos\varphi\sqrt{X^2+Y^2}}\right]+\tan^{-1}\frac{Y}{X} \tag{15}$$

Hence, we have completed the derivation of a function that identifies the longitude of a point on a BTO ring given a latitude in the domain of (15), together with the height *h* of the BTO ring above the reference ellipsoid, and satellite position (X,Y,Z), for any reference ellipsoid controlled by parameters a_{RE} and b_{RE}. The height of the BTO ring is equivalent to the aeroplane height.

<u>A note on system 9.</u>

In transforming curvilinear coordinates (λ,φ_g,h) to Cartesian ones (x,y,z) it is necessary to use a system such as:

$$\begin{bmatrix}\left(\frac{a_{RE}}{\sqrt{1-e_{RE}^2\sin^2\varphi_g}}+h\right)\cos\varphi_g\cos\lambda\\ \left(\frac{a_{RE}}{\sqrt{1-e_{RE}^2\sin^2\varphi_g}}+h\right)\cos\varphi_g\sin\lambda\\ \left(\frac{a_{RE}(1-e_{RE}^2)}{\sqrt{1-e_{RE}^2\sin^2\varphi_g}}+h\right)\sin\varphi_g\end{bmatrix}=\begin{bmatrix}x\\y\\z\end{bmatrix} \tag{A1}$$

We used a subtly similar but not identical system in (9) above to introduce curvilinear variables in place of Cartesian ones. In (A1) the parameters *a* and *e* are kept fixed, rather than being treated as functions of *h,* as indicated by subscript RE. Under these circumstances the particular name given to latitude φ is *geodetic latitude*, and it is a measure of the angle made with the equatorial plane by line erected normal to the ellipsoid surface. Such a line provides a convenient and natural choice for a third axis, completing the orthogonal system (λ,φ_g,h), and it is almost (but not exactly) along this line that a local gravity vector acts.

In our problem we needed to find the intersection of an ellipsoid and a sphere which required us to alter the size of the reference ellipsoid chosen to represent the earth's surface. We were only interested in describing a set of points, the closed BTO ring, on the surface of a modified, or enlarged, ellipsoid. The enlarged ellipsoid was defined by adding the height of the aeroplane, equivalent to the height of the BTO ring, above the reference ellipsoid to the semiaxes of that ellipsoid. In taking this approach we required a slightly different transformation than the one presented in (A1). Our transformation, given in system (9), is equivalent to:

$$\left(\frac{a_{RE}+h}{\sqrt{1-\left(1-\left(\frac{b_{RE}+h}{a_{RE}+h}\right)^2\right)\sin^2\varphi_a}}\right)\begin{bmatrix}\cos\varphi_a\cos\lambda\\ \cos\varphi_a\sin\lambda\\ \left(\frac{b_{RE}+h}{a_{RE}+h}\right)^2\sin\varphi_a\end{bmatrix}=\begin{bmatrix}x\\y\\z\end{bmatrix} \tag{A2}$$

In system (A2) the latitude φ has been subscripted with an *a* to denote an *aerodetic latitude*. Similarly in system (A1) the latitude is subscripted with a *g*, denoting *geodetic latitude*. It is important to realise that when $h\neq 0$ these two latitudes will not be equal for a given Cartesian position (x,y,z). The inequality of these two latitudes can be appreciated by realising that a line erect normal to the reference ellipsoid will not also be normal to the surface of the modified ellipsoid.

In (15) we obtained the longitude λ of a position on a BTO ring given (1) a satellite position (X,Y,Z), (2) an aeroplane-satellite distance *R*, (3) the height *h* of the aeroplane above the reference ellipsoid (measured normal to its surface), and (4) the latitude φ of the aeroplane. What was left unstated however was that the latitude φ is strictly what we call the *aerodetic latitude*, φ_a, rather than the more familiar geodetic latitude. In presenting a projection of the BTO ring it will inevitably be desired to depict the position of the ring with respect to the reference ellipsoid rather than the enlarged one. There are a couple of proposed solutions to this difficulty.

First, the numerical difference between φ_a and φ_g for all *h* attainable by modern transport aeroplanes is approximately equivalent to an arc length on the reference ellipsoid surface to the order of centimetres in the worst case.

Second, although no closed-form transformation exists to map (λ, φ_a, h) to (λ, φ_g, h) there are several available efficient numerical methods for mapping (x,y,z) to (λ, φ_g, h); the Cartesian vector in this case can be obtained by closed-form transformation of (λ, φ_a, h) using (A2). A recommended method based on Hailey's method is presented in (Fukushima, 1999 & 2006)[2].

In conclusion we may neglect the difference between geodetic and aerodetic latitudes for practical work on this problem, but numerical methods presented in Fukushima can be used to obtain precision to the order of arcmicroseconds.

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[2] Fukushima T., (1999): Fast transform from geocentric to geodetic coordinates, Journal of Geodesy, Vol. 73, pp. 603–610.
Fukushima T., (2006): Transformation from Cartesian to geodetic coordinates accelerated by Halley's method, Journal of Geodesy, Vol. 79, pp. 689–693.