1 Analytic Fuel Flow Model

1.1 Overview

To determine both the range and endurance for the aircraft cruising at a constant flight level we identify functions describing the fuel flow rate and speed profile. In this analysis we are primarily interested in the long range cruise (LRC) mode although the M0.84 constant Mach cruise mode is also considered. Solutions for endurance are presented in section 1.2, followed in section 1.3 by range solutions. In subsection 1.2.1 we consider an endurance solution for the LRC mode based on a linear fuel flow rate model; later in subsection 1.2.2 a quadratic fuel flow rate function is introduced to identify the constant Mach endurance solution.

1.2 Endurance

1.2.1 LRC Endurance

Let dm/dt be the fuel flow rate and m the aeroplane mass, then in a standard atmosphere[1]

$$\frac{dm}{dt} = A_0 + A_1 m \tag{1.1}$$

The fuel flow rate varies linearly with deviation in total temperature from its standard value. At speeds below the hypersonic regime total temperature T_0, also known as the stagnation temperature, is a function of the Mach number M and static temperature T_s alone[2]

$$T_0 = T_s\left(1 + \frac{\gamma - 1}{2} M^2\right) \tag{1.2}$$

where the constant $\gamma = 7/5$ is the adiabatic index for air. The static temperature may be considered as the sum of the standard static temperature T_{ISA} and a static temperature deviation term ΔT_s

$$T_s = T_{ISA} + \Delta T_s \tag{1.3}$$

[1]International Organization for Standardization, *Standard Atmosphere*, ISO 2533:1975, 1975.

[2]Shapiro, A. H. (1953). The Dynamics and Thermodynamics of Compressible Fluid Flow. Vol. I. New York,The Ronald Press Company.

in turn the standard static temperature is a linear function of the standard pressure altitude h in the troposphere but remains constant in the lower stratosphere

$$T_{ISA} = \begin{cases} T_{SL} - h\Gamma & \text{for } [h \leq 11\,000\,\text{metres}] \\ T_{SL} - 11000\Gamma & \text{for } [h > 11\,000\,\text{metres}] \end{cases} \tag{1.4}$$

where $T_{SL} = 288.15$ K is the standard sea level temperature and $\Gamma = 0.0065$ K m^{-1} is the temperature lapse rate.

Total temperature deviation, ΔT_0in terms of static temperature deviation is then

$$\Delta T_0 = \Delta T_s(1 + \frac{\gamma - 1}{2} M^2) \tag{1.5}$$

and since the fuel flow rate varies linearly with total temperature deviation, we may write

$$\frac{dm}{dt} = (A_0 + A_1 m)\,(1 + k_2 \Delta T_0) \tag{1.6}$$

which, using (1.5) in (1.6), is equivalent to

$$\frac{dm}{dt} = (A_0 + A_1 m) \left(1 + k_2 \Delta T_s(1 + \frac{\gamma - 1}{2} M^2)\right) \tag{1.7}$$

where the constant $k_2 = {}^3\!/_{1000}$ is a fuel flow rate adjustment parameter to account for non-standard temperature.

At a given cruise flight level the square of the Mach number in (1.7) is a function of the aeroplane all up mass

$$M^2 = C + Dm + Em^2 \tag{1.8}$$

The LRC fuel flow rate in (1.7) after some manipulations is written as

$$\frac{dm}{dt} = (A_0 + A_1 m) \left(F + 2Gm + Hm^2\right) \tag{1.9}$$

$$F = 1 + k_2 \Delta T_s(1 + \frac{\gamma - 1}{2} C), \quad G = \tfrac{\gamma-1}{4} k_2 \Delta T_s D, \quad H = \frac{\gamma - 1}{2} k_2 \Delta T_s E \tag{1.10}$$

After separating the variables and integrating (1.9) yields the endurance[3] t

$$t = \int_{m_0}^{m_1} \frac{dm}{(A_0 + A_1 m)\,(F + 2Gm + Hm^2)} \tag{1.11}$$

[3]Endurance in this context means the total time in the cruise, viz., the time from the top of climb until fuel exhaustion.

Decomposing the integrand in (1.11) into partial fractions

$$\frac{1}{(A_0 + A_1 m)\,(F + 2Gm + Hm^2)} \equiv K\left(\frac{1}{A_0 + A_1 m} + \frac{Nm + P}{F + 2Gm + Hm^2}\right) \quad (1.12)$$

$$K = \frac{A_1^2}{A_0^2 H - 2A_0 A_1 G + A_1^2 F}, \quad N = -\tfrac{H}{A_1}, \quad P = \frac{A_0 H - 2A_1 G}{A_1^2}$$

leading to

$$t = K\int_{m_0}^{m_1} \frac{dm}{A_0 + A_1 m} + K\int_{m_0}^{m_1} \frac{(Nm + P)\,dm}{F + 2Gm + Hm^2} \quad (1.13)$$

The first integral in (1.13) evaluates to

$$K\int_{m_0}^{m_1} \frac{dm}{A_0 + A_1 m} = \frac{K}{A_1} \ln|A_0 + A_1 m|\Big|_{m_0}^{m_1} \quad (1.14)$$

while the second[4] evaluates to, omitting the limits for clarity

$$K\int_{m_0}^{m_1} \frac{(Nm + P)\,dm}{F + 2Gm + Hm^2} \quad (1.15)$$

$$= K\left(\frac{N}{2H} \ln\left|F + 2Gm + Hm^2\right| + \frac{PH - NG}{H\sqrt{FH - G^2}} \arctan \frac{Hm + G}{\sqrt{FH - G^2}}\right) \text{for } \left[FH > G^2\right]$$

$$= K\left(\frac{N}{2H} \ln\left|F + 2Gm + Hm^2\right| + \frac{PH - NG}{2H\sqrt{G^2 - FH}} \ln\left|\frac{Hm + G - \sqrt{G^2 - FH}}{Hm + G + \sqrt{G^2 - FH}}\right|\right) \text{for } \left[FH < G^2\right]$$

If $\Delta T_s = 0$ then $FH = G^2$ for which neither case in (1.15) is defined. The three endurance solutions are

$$t = \begin{cases} \frac{1}{A_1} \ln|A_0 + A_1 m|\Big|_{m_0}^{m_1} & \text{for } [\Delta T_s = 0] \\ \frac{K}{A_1}\left(\frac{1}{2}\ln\left|\frac{(A_0+A_1 m)^2}{F+2Gm+Hm^2}\right| + \frac{A_0 H - A_1 G}{A_1\sqrt{FH-G^2}} \arctan \frac{Hm+G}{\sqrt{FH-G^2}}\right)\Big|_{m_0}^{m_1} & \text{for } [FH > G^2] \\ \frac{K}{A_1}\left(\frac{1}{2}\ln\left|\frac{(A_0+A_1 m)^2}{F+2Gm+Hm^2}\right| + \frac{A_0 H - A_1 G}{2A_1\sqrt{G^2-FH}} \ln\left|\frac{Hm+G-\sqrt{G^2-FH}}{Hm+G+\sqrt{G^2-FH}}\right|\right)\Big|_{m_0}^{m_1} & \text{for } [FH < G^2] \end{cases} \quad (1.16)$$

To model performance degradation through an increased fuel flow rate simply scale coefficients A_0 and A_1 equally as required.

[4]Gradshteyn, I.S. and Ryzhik, I.M. (2007). Table of Integrals, Series, and Products. 7th Ed. Section 2.103(5). London: Elsevier.

1.2.2 Mach 0.84 Cruise Endurance

Let dm/dt be the fuel flow rate and m the aeroplane mass, then in a standard atmosphere

$$\frac{dm}{dt} = B_0 + 2B_1 m + B_2 m^2 \tag{1.17}$$

The fuel flow rate for the Mach 0.84 cruise also varies linearly with total temperature deviation. Replacing the first factor in (1.7) with (1.17)

$$\frac{dm}{dt} = \left(B_0 + 2B_1 m + B_2 m^2\right)\left(1 + k_2 \Delta T_s (1 + \frac{\gamma - 1}{2} M^2)\right) \tag{1.18}$$

where once again the constant $k_2 = 3/1000$ is a fuel flow rate adjustment parameter to account for non-standard temperature and the constant adiabatic index is $\gamma = 7/5$. The Mach number in (1.18) is independent of the aeroplane mass. Separating the variables and integrating to obtain the endurance

$$t = \frac{1}{1 + k_2 \Delta T_s (1 + \frac{\gamma-1}{2} M^2)} \int_{m_0}^{m_1} \frac{dm}{B_0 + 2B_1 m + B_2 m^2} \tag{1.19}$$

The integral in (1.19) has two solutions, omitting the limits for clarity

$$\int_{m_0}^{m_1} \frac{dm}{B_0 + 2B_1 m + B_2 m^2} = \begin{cases} \frac{1}{\sqrt{B_0 B_2 - B_1^2}} \arctan \frac{B_2 m + B_1}{\sqrt{B_0 B_2 - B_1^2}} & \text{for } [B_0 B_2 > B_1^2] \\ \frac{1}{2\sqrt{B_1^2 - B_0 B_2}} \ln \left| \frac{B_2 m + B_1 - \sqrt{B_1^2 - B_0 B_2}}{B_2 m + B_1 + \sqrt{B_1^2 - B_0 B_2}} \right| & \text{for } [B_0 B_2 < B_1^2] \end{cases} \tag{1.20}$$

leading to the two endurance solutions, for $M = 0.84$,

$$t = \begin{cases} \frac{\left(1 + k_2 \Delta T_s (1 + \frac{\gamma-1}{2} M^2)\right)^{-1}}{\sqrt{B_0 B_2 - B_1^2}} \arctan \frac{B_2 m + B_1}{\sqrt{B_0 B_2 - B_1^2}} & \text{for } [B_0 B_2 > B_1^2] \\ \frac{\left(1 + k_2 \Delta T_s (1 + \frac{\gamma-1}{2} M^2)\right)^{-1}}{2\sqrt{B_1^2 - B_0 B_2}} \ln \left| \frac{B_2 m + B_1 - \sqrt{B_1^2 - B_0 B_2}}{B_2 m + B_1 + \sqrt{B_1^2 - B_0 B_2}} \right| & \text{for } [B_0 B_2 < B_1^2] \end{cases} \tag{1.21}$$

1.3 Range

In this subsection *range* means the air distance the aeroplane can travel at a fixed flight level in either LRC or constant Mach cruise given a specified fuel quantity, an initial or final aeroplane mass and an assumed uniform temperature deviation.

1.3.1 LRC Range

In an LRC cruise the Mach number depends on the aeroplane mass and is treated as being independent of temperature deviation. The speed of the aeroplane with respect to the surrounding air mass is related to the Mach number by the speed of sound which depends on the air mass temperature T_s. Let ds/dt be the speed, M the Mach number and c_s the local speed of sound

$$\frac{ds}{dt} = Mc_s \tag{1.22}$$

where $c_s = \sqrt{\gamma R T_s}$, $\gamma = 7/5$ is the adiabatic index and R is the gas constant for air.[5] Letting the LRC Mach number be a quadratic function of mass

$$M = C_2 + D_2 m + E_2 m^2 \tag{1.23}$$

then the LRC speed is a function of mass and static temperature

$$\frac{ds}{dt} = \left(C_2 + D_2 m + E_2 m^2\right)\sqrt{\gamma R T_s} \tag{1.24}$$

Recalling the LRC fuel flow rate from (1.9)

$$\frac{dm}{dt} = (A_0 + A_1 m)\left(F + 2Gm + Hm^2\right)$$

and using the chain rule, the rate of change of path length with respect to mass is the rational function

$$\frac{ds}{dt}\frac{dt}{dm} = \sqrt{\gamma R T_s}\frac{C_2 + D_2 m + E_2 m^2}{(A_0 + A_1 m)\left(F + 2Gm + Hm^2\right)} \tag{1.25}$$

We now decompose (1.25) into partial fractions

$$\sqrt{\gamma R T_s}\frac{C_2 + D_2 m + E_2 m^2}{(A_0 + A_1 m)\left(F + 2Gm + Hm^2\right)} = K_2\sqrt{\gamma R T_s}\left(\frac{1}{A_0 + A_1 m} + \frac{N_2 m + P_2}{F + 2Gm + Hm^2}\right)$$

[5] The adiabatic index is constant below approximately Mach 3 in the troposphere and stratosphere, and the gas constant remains constant at all altitudes reachable by transonic commercial aeroplanes.

(1.26)

$$K_2 = \frac{A_0^2 E_2 - A_0 A_1 D_2 + A_1^2 C_2}{A_0^2 H - 2A_0 A_1 G + A_1^2 F}, \quad N_2 = \frac{A_0 D_2 H - 2A_0 E_2 G - A_1 C_2 H + A_1 E_2 F}{A_0^2 E_2 - A_0 A_1 D_2 + A_1^2 C_2}, \quad P_2 = \frac{A_0 C_2 H - 2A_1 C_2 G - A_0 E_2 F + A_1 D_2 F}{A_0^2 E_2 - A_0 A_1 D_2 + A_1^2 C_2}$$

Using the right member of (1.26) in (1.25), separating the variables and integrating yields the range potential s

$$s = K_2 \sqrt{\gamma R T_s} \int_{m_0}^{m_1} \frac{dm}{A_0 + A_1 m} + K_2 \sqrt{\gamma R T_s} \int_{m_0}^{m_1} \frac{(N_2 m + P_2) dm}{F + 2Gm + Hm^2} \tag{1.27}$$

The first integral in (1.27) evaluates to

$$K_2 \sqrt{\gamma R T_s} \int_{m_0}^{m_1} \frac{dm}{A_0 + A_1 m} = \frac{K_2}{A_1} \sqrt{\gamma R T_s} \ln |A_0 + A_1 m| \Big|_{m_0}^{m_1} \tag{1.28}$$

while the second integral has two solutions, omitting the limits for clarity

$$\begin{aligned} & K_2 \sqrt{\gamma R T_s} \int_{m_0}^{m_1} \frac{(N_2 m + P_2)\, dm}{F + 2Gm + Hm^2} \\ & = K_2 \sqrt{\gamma R T_s} \left(\frac{N_2}{2H} \ln \left| F + 2Gm + Hm^2 \right| + \frac{P_2 H - N_2 G}{H\sqrt{FH - G^2}} \arctan \frac{Hm + G}{\sqrt{FH - G^2}} \right) \\ & \text{for } \left[FH > G^2 \right], \\ & = K_2 \sqrt{\gamma R T_s} \left(\frac{N_2}{2H} \ln \left| F + 2Gm + Hm^2 \right| + \frac{P_2 H - N_2 G}{2H\sqrt{G^2 - FH}} \ln \left| \frac{Hm + G - \sqrt{G^2 - FH}}{Hm + G + \sqrt{G^2 - FH}} \right| \right) \\ & \text{for } \left[FH < G^2 \right] \end{aligned} \tag{1.29}$$

Analogous to (1.15), when $\Delta T_s = 0$ then $FH = G^2$ for which neither case in **??** is defined. In this case we must rewrite (1.25) using the fuel flow rate given in (1.1) instead of (1.9). Recalling (1.1) for the fuel flow rate applicable when $\Delta T_s = 0$,

$$\frac{dm}{dt} = A_0 + A_1 m$$

leads to the rate of change of path length with mass, in a standard atmosphere, becoming

$$\frac{ds}{dt} \frac{dt}{dm} = \sqrt{\gamma R T_s} \frac{C_2 + D_2 m + E_2 m^2}{A_0 + A_1 m} \tag{1.30}$$

and after separating into an integral part and remainder is

$$\frac{ds}{dm} = \frac{\sqrt{\gamma R T_s}}{A_1^2}(A_1 E_2 m + A_1 D_2 - A_0 E_2 + \frac{A_0^2 E_2 - A_0 A_1 D_2 + A_1^2 C_2}{A_0 + A_1 m}) \tag{1.31}$$

in turn leading to the standard atmosphere range potential

$$s = \frac{\sqrt{\gamma R T_s}}{2A_1^2}(A_1^2 E_2 m^2 + 2(A_1 D_2 - A_0 E_2)m + \frac{2}{A_1}(A_0^2 E_2 - A_0 A_1 D_2 + A_1^2 C_2)\ln|A_0 + A_1 m|)\Bigg|_{m_0}^{m_1} \tag{1.32}$$

Altogether the three LRC range solutions are

$$s = \begin{cases} \frac{\sqrt{\gamma R T_s}}{2A_1^2}(A_1^2 E_2 m^2 + 2(A_1 D_2 - A_0 E_2)m + \frac{2}{A_1}(A_0^2 E_2 - A_0 A_1 D_2 + A_1^2 C_2)\ln|A_0 + A_1 m|)\Big|_{m_0}^{m_1} \\ \text{for } [\Delta T_s = 0]\,, \\ \\ K_2\sqrt{\gamma R T_s}\left(\frac{N_2}{2H}\ln|F + 2Gm + Hm^2| + \frac{P_2 H - N_2 G}{H\sqrt{FH - G^2}}\arctan\frac{Hm + G}{\sqrt{FH - G^2}}\right)\Big|_{m_0}^{m_1} \\ \text{for } \ [FH > G^2]\,, \\ \\ K_2\sqrt{\gamma R T_s}\left(\frac{N_2}{2H}\ln|F + 2Gm + Hm^2| + \frac{P_2 H - N_2 G}{2H\sqrt{G^2 - FH}}\ln\left|\frac{Hm + G - \sqrt{G^2 - FH}}{Hm + G + \sqrt{G^2 - FH}}\right|\right)\Big|_{m_0}^{m_1} \\ \text{for } \ [FH < G^2] \end{cases} \tag{1.33}$$

1.3.2 Mach 0.84 Cruise Range

Recalling (1.21) for the Mach 0.84 cruise endurance

$$t = \begin{cases} \frac{\left(1 + k_2 \Delta T_s (1 + \frac{\gamma - 1}{2} M^2)\right)^{-1}}{\sqrt{B_0 B_2 - B_1^2}} \arctan\frac{B_2 m + B_1}{\sqrt{B_0 B_2 - B_1^2}}\Big|_{m_0}^{m_1} & \text{for } \ [B_0 B_2 > B_1^2] \\ \\ \frac{\left(1 + k_2 \Delta T_s (1 + \frac{\gamma - 1}{2} M^2)\right)^{-1}}{2\sqrt{B_1^2 - B_0 B_2}} \ln\left|\frac{B_2 m + B_1 - \sqrt{B_1^2 - B_0 B_2}}{B_2 m + B_1 + \sqrt{B_1^2 - B_0 B_2}}\right|\Big|_{m_0}^{m_1} & \text{for } \ [B_0 B_2 < B_1^2] \end{cases} \tag{1.34}$$

where $k_2 = 3/1000$ and $\gamma = 7/5$.

The speed obtained in a Mach 0.84 cruise is a function of the static temperature T_s alone

$$\frac{ds}{dt} = M c_s = M\sqrt{\gamma R T_s} \tag{1.35}$$

Integrating (1.35) and substituting (1.34) for endurance t yields the range

$$s = \begin{cases} \frac{M\sqrt{\gamma R T_s}}{(1+k_2\Delta T_s(1+\frac{\gamma-1}{2}M^2))\sqrt{B_0B_2-B_1^2}} \arctan \frac{B_2m+B_1}{\sqrt{B_0B_2-B_1^2}}\Big|_{m_0}^{m_1} & \text{for } [B_0B_2 > B_1^2] \\ \frac{M\sqrt{\gamma R T_s}}{2(1+k_2\Delta T_s(1+\frac{\gamma-1}{2}M^2))\sqrt{B_1^2-B_0B_2}} \ln \left|\frac{B_2m+B_1-\sqrt{B_1^2-B_0B_2}}{B_2m+B_1+\sqrt{B_1^2-B_0B_2}}\right|\Big|_{m_0}^{m_1} & \text{for } [B_0B_2 < B_1^2] \end{cases} \tag{1.36}$$

Barry Martin

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