B.1. ECEF velocity vector

We seek the ECEF Cartesian velocity vector parameterised by the familiar geodetic coordinates longitude (λ), latitude (φ), and height (h), taking account of local azimuth (east of true north) (α), speed ($^{ds}/_{dt}$) and climb angle (γ).

The differential equation for a linear element of arc length of a curve on the surface of the ellipsoid is:

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{A.1}$$

When $x = x(\lambda, \varphi, h)$, $y = y(\lambda, \varphi, h)$, $z = z(\lambda, \varphi, h)$, the differentials in (A.1) become:

$$dx = \left(\frac{\partial x}{\partial \varphi} d\varphi + \frac{\partial x}{\partial \lambda} d\lambda + \frac{\partial x}{\partial h} dh\right), dy = \left(\frac{\partial y}{\partial \varphi} d\varphi + \frac{\partial y}{\partial \lambda} d\lambda + \frac{\partial y}{\partial h} dh\right), dz = \left(\frac{\partial z}{\partial \varphi} d\varphi + \frac{\partial z}{\partial \lambda} d\lambda + \frac{\partial z}{\partial h} dh\right) \tag{A.2}$$

Using the squares of the expressions in (A.2) allows (A.1) to be recast as:

$$ds^2 = \left(\frac{\partial x}{\partial \varphi} d\varphi + \frac{\partial x}{\partial \lambda} d\lambda + \frac{\partial x}{\partial h} dh\right)^2 + \left(\frac{\partial y}{\partial \varphi} d\varphi + \frac{\partial y}{\partial \lambda} d\lambda + \frac{\partial y}{\partial h} dh\right)^2 + \left(\frac{\partial z}{\partial \varphi} d\varphi + \frac{\partial z}{\partial \lambda} d\lambda + \frac{\partial z}{\partial h} dh\right)^2 \tag{A.3}$$

Expanding the right members and collecting common factors:

$$\begin{aligned} ds^2 = {} & \underbrace{\left[\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 + \left(\frac{\partial z}{\partial \lambda}\right)^2\right]}_{=\mathrm{E}_1} d\lambda^2 + \underbrace{\left[\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2\right]}_{=\mathrm{E}_2} d\varphi^2 + \underbrace{\left[\left(\frac{\partial x}{\partial h}\right)^2 + \left(\frac{\partial y}{\partial h}\right)^2 + \left(\frac{\partial z}{\partial h}\right)^2\right]}_{=\mathrm{E}_3} dh^2 + \\ & 2\underbrace{\left[\frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial \lambda} + \frac{\partial z}{\partial \varphi}\frac{\partial z}{\partial \lambda}\right]}_{=\mathrm{E}_4} d\varphi d\lambda + 2\underbrace{\left[\frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial h} + \frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial h} + \frac{\partial z}{\partial \varphi}\frac{\partial z}{\partial h}\right]}_{=\mathrm{E}_5} d\varphi dh + 2\underbrace{\left[\frac{\partial x}{\partial \lambda}\frac{\partial x}{\partial h} + \frac{\partial y}{\partial \lambda}\frac{\partial y}{\partial h} + \frac{\partial z}{\partial \lambda}\frac{\partial z}{\partial h}\right]}_{=\mathrm{E}_6} d\lambda dh \end{aligned} \tag{A.4}$$

The quantities (E_4 ... E_6) = 0 because the geodetic coordinate axes are orthogonal. The linear element in (A.4) is therefore simply:

$$ds^2 = E_1 d\lambda^2 + E_2 d\varphi^2 + E_3 dh^2 = \left\|\frac{\partial \vec{\mathbf{x}}}{\partial \lambda}\right\|^2 d\lambda^2 + \left\|\frac{\partial \vec{\mathbf{x}}}{\partial \varphi}\right\|^2 d\varphi^2 + \left\|\frac{\partial \vec{\mathbf{x}}}{\partial \mathrm{h}}\right\|^2 dh^2 \tag{A.5}$$

The quantities (E_1 ... E_3) can be found from the Jacobian matrix of the transformation $\mathrm{F}(\vec{\boldsymbol{\lambda}})$. The transformation from geodetic to Cartesian coordinates, $\vec{\mathbf{x}} = \mathrm{F}(\vec{\boldsymbol{\lambda}})$, where $\vec{\mathbf{x}}$ is the 3-element ECEF Cartesian position vector, and $\vec{\boldsymbol{\lambda}}$ is the 3-element geodetic position vector, is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (N(\varphi) + h)\cos\varphi\cos\lambda \\ (N(\varphi) + h)\cos\varphi\sin\lambda \\ \left((1 - e^2)N(\varphi) + h\right)\sin\varphi \end{bmatrix} \tag{A.6}$$

where *e* is the first elliptic eccentricity of the ellipsoid, and function *N*(ϕ) is the distance between a point on the surface and the rotational axis measured along a normal to the surface. It is variously described as the radius of curvature in the prime vertical and the transverse radius of curvature:

$$N(\varphi) = \frac{a}{\sqrt{1-e^2 \sin^2 \varphi}} \tag{A.7}$$

which contains the ellipsoid parameter *a* for the semimajor axis.

The Jacbobian matrix for the transformation $\mathrm{F}(\vec{\boldsymbol{\lambda}})$ in (A.6) is:

$$\mathbf{J}_F(\lambda, \varphi, h) = \begin{bmatrix} -(N(\varphi)+h)\cos\varphi\sin\lambda & -(R(\varphi)+h)\sin\varphi\cos\lambda & \cos\varphi\cos\lambda \\ (N(\varphi)+h)\cos\varphi\cos\lambda & -(R(\varphi)+h)\sin\varphi\sin\lambda & \cos\varphi\sin\lambda \\ 0 & (R(\varphi)+h)\cos\varphi & \sin\varphi \end{bmatrix} \tag{A.8}$$

where function *R*(φ), for the radius of curvature of a meridional ellipse, is:

$$R(\varphi) = \frac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{\frac{3}{2}}} \tag{A.9}$$

The Jacobian matrix in (A.8) can be decomposed into two orthogonal rotation matrices and a diagonal matrix:

$$\mathbf{J}_F(\lambda, \varphi, h) = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ \cos\lambda & \sin\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sin\varphi & \cos\varphi \\ 0 & \cos\varphi & \sin\varphi \end{bmatrix} \begin{bmatrix} (N(\varphi)+h)\cos\varphi & 0 & 0 \\ 0 & (R(\varphi)+h) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A.10}$$

By using the chain rule we can form the Cartesian velocity vector:

$$\frac{d\vec{\mathbf{x}}}{dt} = \frac{\partial\vec{\mathbf{x}}}{\partial\vec{\boldsymbol{\lambda}}}\frac{d\vec{\boldsymbol{\lambda}}}{dt} = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ \cos\lambda & \sin\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sin\varphi & \cos\varphi \\ 0 & \cos\varphi & \sin\varphi \end{bmatrix} \begin{bmatrix} (N(\varphi)+h)\cos\varphi & 0 & 0 \\ 0 & (R(\varphi)+h) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\lambda} \\ \dot{\varphi} \\ \dot{h} \end{bmatrix} \tag{A.11}$$

The aircraft speed ${}^{ds}/_{dt}$ may be introduced to (A.11) by further use of the chain rule:

$$\frac{d\vec{\boldsymbol{\lambda}}}{dt} = \frac{d\vec{\boldsymbol{\lambda}}}{ds}\frac{ds}{dt} \tag{A.12}$$

The linear element in (A.5) can now be expressed in terms of the squares of the entries in the diagonal matrix given in (A.10):

$$ds^2 = (N(\varphi)+h)^2\cos^2\varphi\, d\lambda^2 + (R(\varphi)+h)^2 d\varphi^2 + dh^2 \tag{A.13}$$

The linear element *ds* may be visualised as a vector rotated through azimuth *α* (east of true north) in the local tangent plane and further rotated through climb angle γ in the vertical plane orthogonal to the tangent plane. The geodetic components of the linear element are:

$$\begin{bmatrix} \cos\gamma\sin\alpha\, ds \\ \cos\gamma\cos\alpha\, ds \\ \sin\gamma\, ds \end{bmatrix} = \begin{bmatrix} (N(\varphi)+h)\cos\varphi\, d\lambda \\ (R(\varphi)+h)\, d\varphi \\ dh \end{bmatrix} \tag{A.14}$$

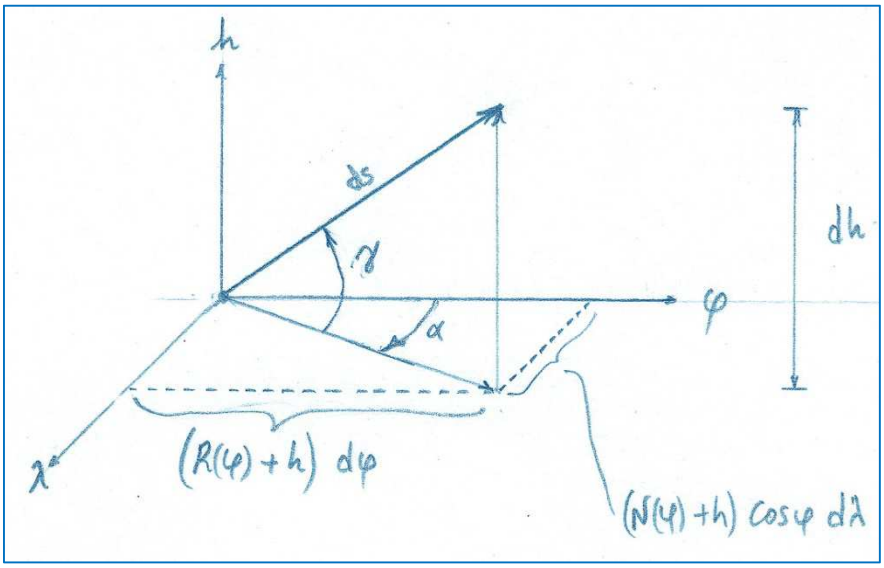

Figure 1. A view of the linear element in the local tangent plane.

The vector equation in (A.14) after rearranging to set $d\vec{\boldsymbol{\lambda}}/_{ds}$ as the subject is:

$$\frac{d\vec{\boldsymbol{\lambda}}}{ds} = \begin{bmatrix} \frac{\cos\gamma \sin\alpha}{(N(\varphi)+h)\cos\varphi} \\ \frac{\cos\gamma\cos\alpha}{(R(\varphi)+h)} \\ \sin\gamma \end{bmatrix} \tag{A.15}$$

So that the Cartesian velocity vector $d\vec{\mathbf{x}}/_{dt}$ becomes:

$$\frac{d\vec{\mathbf{x}}}{dt} = \frac{\partial\vec{\mathbf{x}}}{\partial\vec{\boldsymbol{\lambda}}}\frac{d\vec{\boldsymbol{\lambda}}}{ds}\frac{ds}{\mathrm{d}t} \tag{A.16}$$

$$\frac{d\vec{\mathbf{x}}}{dt} = \frac{ds}{dt}\begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ \cos\lambda & \sin\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sin\varphi & \cos\varphi \\ 0 & \cos\varphi & \sin\varphi \end{bmatrix}\begin{bmatrix} \cos\gamma\sin\alpha \\ \cos\gamma\cos\alpha \\ \sin\gamma \end{bmatrix} \tag{A.17}$$

The scalar $ds/_{dt}$ in (A.17) is the speed of the aircraft along its trajectory which is the same as the norm of its velocity vector. Under this interpretation, the rate of climb is $ds/_{dt}\sin\gamma$ and the horizontal speed, which is strictly only 'ground speed' when *h* is zero, is $ds/_{dt}\cos\gamma$.

Barry Martin
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w3@aqqa.org